Robotics

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1 Robot kinematics

Robotics is an interdisciplinary field involving mechanical and electrical engineering, computer science, and applied mathematics, that also draws inspirations from biology, psychology, and cognitive science. From robots in a factory to interplanetary rovers, one of the fundamental capabilities a robot must have is to know where it is and how to get to where it needs to go. This includes knowing where its appendages are and how to move them effectively. While in technical jargon these appendages may be called 'manipulators' with 'end-effectors' that directly interact with workpieces, they are more commonly referred to as robot 'arms' and 'hands,' especially when the robot links are connected end to end in serial fashion. Locating a hand involves finding both its position and its orientation. For example, to pick an object out of an open jar, it is important for the hand to not only reach the position of the object but also to be oriented through the opening of the jar. Orientation is fully specified by a 3×3 orthogonal matrix, sometimes called a rotation matrix or a direction cosine matrix.

It turns out that the geometric motion characteristics, that is, the *kinematics*, of most robots (and most mechanical systems in general) are well modeled by systems of polynomial equations. In particular, this can be seen in the most common element used in mechanism work: the rotational joint. When a series of links are connected end to end by rotational joints, each link turns circles with respect to its neighbors, and circles can be described by polynomial equations. The polynomial nature of the models allows one to apply algebraic geometry to solving kinematics problems.

Consider a serial-link robot arm with rotational joints canted at various angles so that the hand maneuvers in three-dimensional space, not just in one plane. To be more precise, consider first just a single joint and let $\vec{u} \in \mathbb{R}^3$ be a unit vector ($\vec{u} \cdot \vec{u} = 1$) along its joint axis, assumed to be passing through the origin. As illustrated in Figure 1, the rotation of an arbitrary vector $\vec{p} \in \mathbb{R}^3$ through an angle of θ around unit vector $\vec{u}$ is

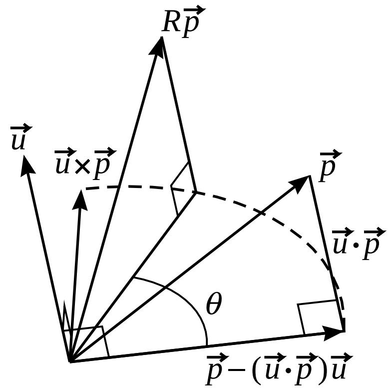

Figure 1: Rotation of $\vec{p}$ around axis $\vec{u}$.

$$R(\vec{u},\theta)\vec{p} = (\vec{u}\cdot\vec{p})\vec{u} + [\vec{p} - (\vec{u}\cdot\vec{p})\vec{u}]\cos\theta + \vec{u}\times\vec{p}\,\sin\theta,$$

where $R(\vec{u},\theta)$ is a 3×3 matrix expression formed from the matrix interpretation of the vectorial operations on the right-hand side.

The trigonometric expression for the rotation can be converted to an algebraic one by replacing $(\cos\theta, \sin\theta)$ by (c, s) subject to the unit-circle condition $c^2 + s^2 = 1$. Abusing notation, we call the reformulated rotation matrix $R(\vec{u}, c, s)$, which in matrix form becomes

$$R(\vec{u}, c, s) = \vec{u}\vec{u}^T + (I - \vec{u}\vec{u}^T)c + \Lambda(\vec{u})s,$$

where

$$\Lambda\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}.$$

For example, if $\vec{u} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, then $R(\vec{u}, c, s)$ is a rotation in the (y, z)-plane. We note that $R(\vec{u}, c, s)$ is linear in (c, s). As the joint turns, the vector $\vec{u}$ defining it stays constant while (c, s) vary, but $R(\vec{u}, c, s)$ is always orthogonal, and its determinant is 1.

This generalizes to a multilink arm. Consider links numbered 0 to N from the robot's base to its hand, connected in series by rotational joints. To formulate the position and orientation of the hand with respect to the base, we need geometric information about the links and their joints. As

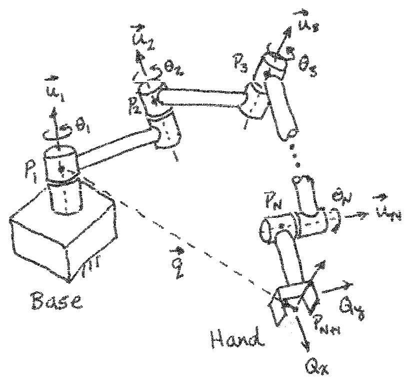

Figure 2: Serial-link robot schematic.

illustrated in Figure 2, mark a point P_1 on the joint axis between links 0 and 1, and similarly mark points $P_2, \ldots, P_N$ on the succeeding axes. Also, mark a reference point in the hand as P_{N+1}. Next, freeze the arm in some initial pose, and at this configuration let $\vec{u}_i$ be a unit vector along the axis of the joint between link $i-1$ and link i. Finally, in the initial pose, let the vector from P_i to P_{i+1} be $\vec{p}_i$ and let the initial orientation of the hand be $Q_0 \in SO(3)$. With these definitions and the shorthand $R_i := R(\vec{u}_i, c_i, s_i)$, we may write the orientation, $Q = \begin{bmatrix} Q_x & Q_y & Q_z \end{bmatrix} \in SO(3)$, of the hand with respect to the base and the position vector, $\vec{q}$, from P_1 to P_N as

$$Q = R_1 R_2 \cdots R_{N-1} R_N Q_0, \tag{1}$$

$$\vec{q} = R_1(\vec{p}_1 + R_2(\vec{p}_2 + \cdots + R_N(\vec{p}_N)\cdots)). \tag{2}$$

Given the joint rotations (c_i, s_i), $i = 1, \ldots, N$, one may evaluate these expressions to obtain Q and $\vec{q}$. This solves the *forward kinematics problem* for serial-link arms.

2 Inverse kinematics

Evaluation of the forward kinematics formulas tells where a serial-link robot's hand is relative to its base. More challenging is to reverse this by answering the *inverse kinematics problem*: what joint rotations (c_i, s_i), $i = 1, \ldots, N$ will cause the hand to attain a desired location $(Q, \vec{q})$? The space of all rigid body motions is $SE(3)$, a six dimensional space parameterizable by three rotations and three translations. Thus, we need $N \geq 6$ joints to place the hand in any arbitrary position and orientation within the working volume of the arm.

Let us consider the important case of $N = 6$, where we expect to have a finite number of solutions to the equations (1,2) along with

$$c_i^2 + s_i^2 = 1, \quad i = 1, \ldots, 6. \tag{3}$$

Although (1) has nine entries, only three are independent since $Q^T Q = I$. The isolated solutions of the system are preserved if one takes three random linear combinations of these nine, which with the rest of the equations makes a system of 12 polynomials in 12 unknowns. By Bézout's Theorem, the number of isolated solutions of a system of N polynomials in N unknowns cannot exceed the total degree, defined as the product of the degrees of the equations. For the system at hand, this comes to $6^6 2^6 = 2,985,984$. It turns out that this upper bound is rather loose.

The root count can be reduced by algebraically manipulating the equations. First, using the fact that $R_i^{-1} = R_i^T$, one may rewrite (1,2) as

$$R_3^T R_2^T R_1^T Q = R_4 R_5 R_6 Q_0, \tag{4}$$

$$R_3^T(R_2^T(R_1^T \vec{q} - \vec{p}_1) - \vec{p}_2) = \vec{p}_3 + R_4(\vec{p}_4 + R_5(\vec{p}_5 + R_6 \vec{p}_6)) \tag{5}$$

These equations are now cubic, reducing the total degree to $3^6 2^6 = 46,656$. But the equations are far from being general cubics, because the (c_i, s_i) pairs each appear linearly. Grouping the unknowns into three groups as $\{c_1, s_1, c_4, s_4\}$, $\{c_2, s_2, c_5, s_5\}$, and $\{c_3, s_3, c_6, s_6\}$, equations (4,5) are recognized as being trilinear. A three-homogeneous variant of Bézout's Theorem applies, bounding the maximum possible number of isolated roots by the coefficient of $\alpha^4\beta^4\gamma^4$ in $(\alpha + \beta + \gamma)^6(2\alpha)^2(2\beta)^2(2\gamma)^2$, i.e., 5,760.

This is just the beginning of the algebraic manipulations that one can perform on the way to showing that the six-revolute (6R) inverse kinematic problem has at most 16 isolated roots.

From an early statement of the problem by Pieper in 1968 to a numerical solution by Tsai and Morgan in 1985 using continuation to the first algebraic derivation of an eliminant equation of degree 16 by Lee and Liang in 1988, this problem was one of the top conundrums for kinematicians for twenty years. Now, powerful computer algorithms can be applied to solve the problem in minutes with either symbolic computer algebra, based on variants of Buchberger's algorithm, or numerical algebraic geometry, based on continuation methods. In the latter approach, no further manipulation of the equations is required, as one can set up a homotopy that continuously deforms a general three-homogeneous system compatible with equations (3–5) into a target 6R example. The endpoints of the 5,760 paths of this homotopy, which can be tracked in parallel on multiple processors, include the 16 isolated solutions (real and complex) of the example 6R problem. After solving a general target example in this way, it becomes the start system for a 16-path parameter homotopy to solve any other 6R inverse kinematic problem.

3 Generalizations

In addition to serial-link arms, robots and mechanisms can have a variety of topologies, composed of serial chains connected together to form closed-chain loops. Both forward and inverse kinematics problems become challenging, but the kinematics remain algebraic and modern algorithms derived from algebraic geometry apply. These mathematical methods for kinematic chains also find application in biomechanical models of humans and animals and in studies of protein folding.

Further Reading

1. Raghavan, M. and Roth, B., 1995. Solving polynomial systems for the kinematic analysis and synthesis of mechanisms and robot manipulators, *J. Mech. Des.* **117**B, 71–79.
2. Wampler, C.W. and Sommese, A.J. 2011. Numerical algebraic geometry and algebraic kinematics. *Acta Numerica* **20**, 469–567.

Biography of contributor

The author is a Technical Fellow at the General Motors R&D Center. He has a Ph.D. in Mechanical Engineering from Stanford University and is a Fellow of both the American Society of Mechanical Engineers and the Institute of Electrical and Electronics Engineers.